Work, energy and power

The topic of mechanics continues in this chapter with the fundamental concept of work done by a force, as well as the concepts of energy and power. Applications of the law of conservation of energy are discussed.

Objectives

By the end of this chapter you should be able to:

- state the definition of *work done by a force*, $W = Fs\cos\theta$, appreciate the significance of the angle appearing in the formula and understand that this formula can only be used when the force is *constant*;
- understand that the *work done by a varying* force is given by the *area under the graph* of force versus displacement;
- state the definitions of *kinetic energy*, $E_k = \frac{1}{2}mv^2$ (also $E_k = \frac{p^2}{2m}$), *gravitational potential energy*, $E_p = mgh$, and *elastic potential energy*, $E_e = \frac{1}{2}kx^2$.
- appreciate that gravitational potential energy can be calculated by measuring heights from an *arbitrary* level;
- understand that, when frictional forces are absent, the total energy $E = E_k + E_p + E_e = \frac{1}{2}mv^2 + mgh + \frac{1}{2}kx^2$ is conserved;
- use the work–kinetic energy relation that states that the work done by the net force is the *change in kinetic energy*;
- understand that, in the presence of *external forces*, the work done is the change in the mechanical energy, $W \doteq \Delta E$;
- state the definition of *power*, $P = \frac{\Delta W}{\Delta t}$, and its very useful form in mechanics, $P = Fv$;
- calculate the *efficiency* of simple machines;
- understand that in all collisions *momentum is always conserved*, but that *kinetic energy* is only conserved in *elastic* collisions.

Work done by a force

Consider a *constant* force $\vec{F}$ acting on a body of mass m as shown in Figure 7.1. The body moves a distance s along a straight line.

The force is always acting upon the body as it moves. Note that the force moves its point of application by a distance s. We define a quantity called the **work** *done by the force* $\vec{F}$ by

$$W = Fs\cos\theta$$

where θ is the angle between the force and the direction along which the mass moves. (The cosine here can be positive, negative or zero; thus work can be positive, negative or zero. We will see what that means shortly.)

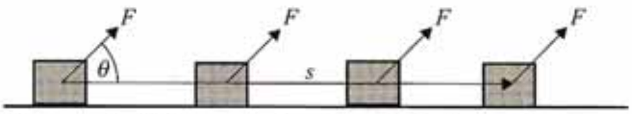

Figure 7.1 A force moving its point of application performs work.

▶ The work done by a force is the product of the force times the distance moved by the object in the direction of the force.

The unit of work is the joule. One joule is the work done by a force of 1 N when it moves a body a distance of 1 m in the direction of the force. 1 J = 1 N m.

Example question

Q1

A mass is being pulled along a level road by a rope attached to it in such a way that the rope makes an angle of 40.0° with the horizontal. The force in the rope is 20.0 N. What is the work done by this force in moving the mass a distance of 8.00 m along the level road?

Answer

Applying the definition of work done, we have

$W = Fs\cos\theta$

where $F = 20.0\,\text{N}$, $s = 8.00\,\text{m}$ and $\theta = 40°$. Thus

$W = 20 \times 8 \times \cos 40°$
$= 123\,\text{J}$

If the force is not constant or the motion does not take place in a straight line, or both, we must be careful. First consider the case of a force of constant magnitude when the motion is not along a straight line.

Example questions

Q2

Find the work done by the tension in a string, as a mass attached to the end of the string performs circular motion (see Figure 7.2).

Answer

This is a case where the force, although constant in magnitude, changes in direction. However, the

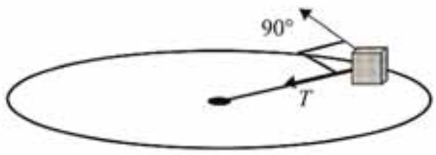

Figure 7.2.

angle between the force and *any small* displacement of the mass as it revolves around the circle is 90° and since cos 90° = 0, the work done is zero.

Q3

A force of constant magnitude 25 N acts on a body that moves along a curved path. The direction of the force is along the velocity vector of the body, i.e. it is *tangential to the path*. Find the work after the mass moves a distance of 50 m along the curved path.

Answer

The diagram on the left of Figure 7.3 shows the path of the body. The diagram on the right is an enlargement of part of the path. We see that the work done when the body travels a *small distance* Δs is

$F\Delta s\cos 0° = F\Delta s$

Breaking up the entire path into small bits in this way, and adding the work done along each bit, we find that the *total* work done is $W = Fs$, where s is the distance travelled along the curved path.

Figure 7.3.

If the force is not constant in magnitude, we must be supplied with the graph that shows the variation of the magnitude of the force with distance travelled. Then we have the following important result:

▶ The area under the graph that shows the variation of the magnitude of the force with distance travelled is the work done.

We can apply this result to the case of the tension in the spring. Since $T = kx$, where k is the spring constant and x the extension (or compression) of the string, the graph of force versus position is as shown in Figure 7.4.

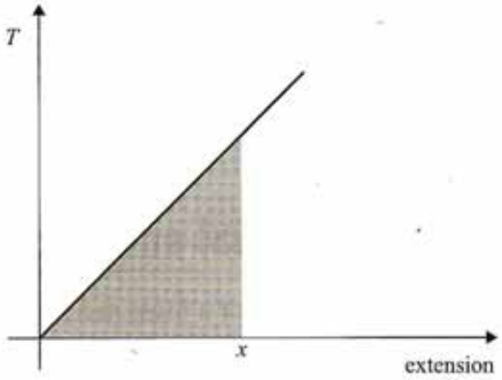

Figure 7.4 The area under a force–distance graph gives the total work performed by the non-constant force.

To find the work done in extending the spring from its natural length ($x = 0$) to extension x, we need to calculate the area of the triangle whose base is x and height is $T = kx$. Thus

$$\begin{aligned} \text{area} &= \tfrac{1}{2}kx \times x \\ &= \tfrac{1}{2}kx^2 \end{aligned}$$

The work to extend a spring from its natural length by an amount x is thus

$$W = \tfrac{1}{2}kx^2$$

It follows that the work done when extending a spring from an extension x_1 to an extension x_2 (so $x_2 > x_1$) is

$$W = \tfrac{1}{2}k\left(x_2^2 - x_1^2\right)$$

Work done by gravity

We will now concentrate on the work done by a very special force, namely the weight of a mass. Remember that weight is mass times acceleration due to gravity and is directed vertically down. Thus, if a mass is displaced horizontally, the work done by mg is zero, since in this case the angle is 90°: $W = mgd\cos 90° = 0$. (See Figure 7.5.)

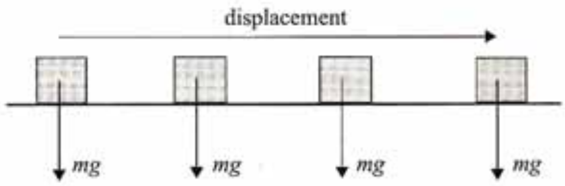

Figure 7.5 The force of gravity is normal to this horizontal displacement so no work is being done.

Note that we are not implying that it is the weight that is forcing the mass to move along the table. We are calculating the work done by a particular force (the weight) if the mass (somehow) moves in a particular way.

If the body falls a vertical distance h, then the work done by W is $+mgh$. The force of gravity is parallel to the displacement, as in Figure 7.6a.

If the mass is thrown vertically upwards to a height h from the launch point, then the work done by W is $-mgh$ since now the angle between direction of force (vertically down) and displacement (vertically up) is 180°. The force of gravity is parallel to the displacement but opposite in direction, as in Figure 7.6b.

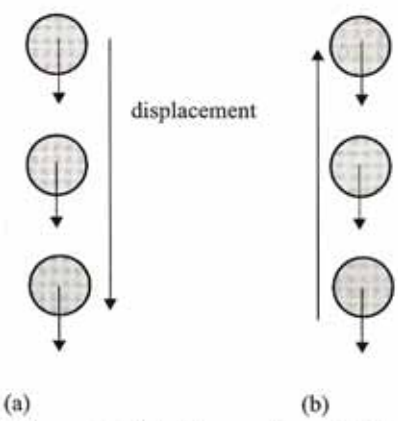

Figure 7.6 The force of gravity is parallel to the displacement in (a) and anti-parallel (i.e. parallel but opposite in direction) in (b).

Consider now the case where a mass moves along some arbitrary path, as shown in Figure 7.7.

Figure 7.7 The work done by gravity is independent of the path followed.

The path consists of horizontal and vertical segments. We now ask about the work done by the weight of the mass. The work done by mg will be equal to the sum of the work done along each horizontal and vertical step. But mg does no work along the horizontal steps since the angle between the force and the displacement in that case will be $90°$ and $\cos 90° = 0$. We are thus left with the vertical steps only. The work done along each step will be $\pm mg\Delta h$, where Δh is the step height. The plus sign is used when we go down a step and the minus sign when we go up a step. (In Figure 7.7 the mass will be forced to go up twice and down eight times.) Thus, what counts is the net number of steps going down (six in our figure). But, this adds up to the vertical distance separating the initial and final position. Hence, the work done by mg is mgh.

If the start and finish positions are joined by an arbitrary smooth curve rather than a 'staircase', the result is still the same. This is because we can always approximate a smooth curve by a series of horizontal and vertical steps; the quality of the approximation depends on how small we take the steps to be. This means that:

▶ The work done by gravity is independent of the path followed and depends only on the vertical distance separating the initial and final positions. The independence of the work done on the path followed is a property of a class of forces (of which weight is a prominent member) called *conservative forces*.

Work done in holding something still

If you try to hold up a heavy object, such as a chair, you will soon get tired. However, the force with which you are holding the chair does zero work since there is no displacement. This is somewhat unexpected. We normally associate getting tired with doing work. Indeed, the forces inside the muscles of the arm and hand holding the chair do work. This is because the muscles stretch and compress and that requires work, just as stretching and compressing a spring does.

Gravitational potential energy

As we just saw, the weight mg of a mass m a height h from the ground will perform work mgh if this mass moves from its position down to the ground. The ability to do this work is there because the mass just happens to be at a height h from the ground. The ability to do work is called energy. When the force in question is the weight (which depends on gravity), we call this energy *gravitational* **potential energy**:

$$E_{\mathrm{p}} = mgh$$

Any mass has gravitational potential energy by virtue of its position. But what determines h? Obviously, we have to choose a reference level from which we will measure heights. But we can choose any level we like. A mass $m = 2\,\mathrm{kg}$ sitting on a table 1 m from the floor will have $E_{\mathrm{p}} = 2 \times 10 \times 1 = 20$ J if the reference level is the floor, but will have $E_{\mathrm{p}} = 0$ if the reference level is the table. If the reference level is chosen to be the ceiling, 2 m above the table, then $E_{\mathrm{p}} = -2 \times 10 \times 2 = -40$ J. (See Figure 7.8.)

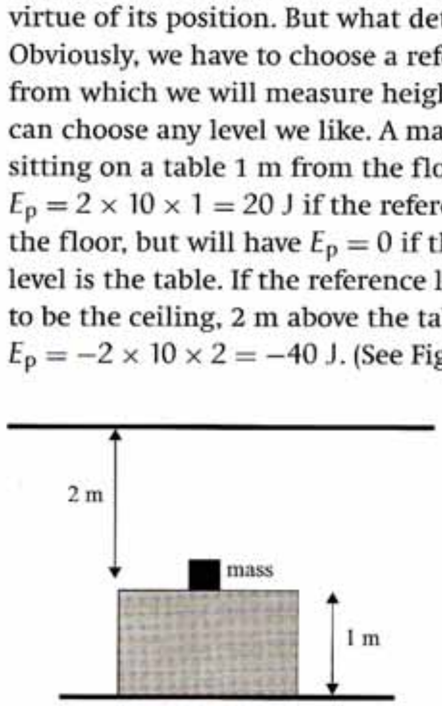

Figure 7.8 The mass has different potential energies depending on the reference level chosen.

So, the same mass will have different gravitational potential energy depending on what reference level we choose. This might seem to make E_p a useless quantity. But if you are patient, you will see that this is not the case.

Potential energy can be understood in the following way. Consider a mass resting on a horizontal floor. If an external force equal to mg is applied to the mass vertically up and the mass moves without acceleration to a position h metres higher than the floor, the work done *by the external force* is mgh. What has become of this work? This work has gone into gravitational potential energy of the mass. This energy is stored as potential energy in the new position of the mass. Similarly, if a spring is initially unstretched and an external force stretches it by an amount x, then the work done by this external force is $\frac{1}{2}kx^2$. This work is stored as elastic potential energy in the (now stretched) spring.

▶ This is a general result for all kinds of potential energies: when an *external* force changes the state of a system without acceleration and does work W in the process, the work so performed is stored as potential energy in the new state of the system.

Example question

Q4

A mass of 10 kg rests on top of a vertical spring whose base is attached to the floor. The spring compresses by 5 cm. What is the spring constant of the spring? How much energy is stored in the spring?

Answer

The mass is at equilibrium so

$$mg = kx$$

$$\Rightarrow k = \frac{mg}{x} = \frac{100}{0.05} = 2000\,\text{N m}^{-1}$$

The stored energy is

$$E_e = \frac{1}{2}kx^2 = \frac{1}{2} \times 2000(0.05)^2 = 2.5\,\text{J}$$

The work–kinetic energy relation

What effect does the work done have on a body? When a body of mass m is acted upon by a *net* force F, then this body experiences an acceleration $a = \frac{F}{m}$ in the direction of F. Suppose that this body had speed v_0 when the force was first applied to it and that the speed after moving a distance x (in the direction of the net force) becomes v_f, as shown in Figure 7.9.

Figure 7.9 A force accelerates a mass, increasing its kinetic energy.

We know from kinematics that

$$v_f^2 = v_0^2 + 2ax$$

so replacing the acceleration by F/m we find

$$v_f^2 = v_0^2 + 2\frac{F}{m}x$$

$$\Rightarrow Fx = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_0^2$$

But Fx is the work done on the mass. This work equals the change in the quantity $E_k = \frac{1}{2}mv^2$, a quantity called the **kinetic energy** of the mass. We thus see that the net work done on a mass results in a change of the kinetic energy of the object. This is a very useful result with applications in many areas of physics.

▶ The work done by the *net* force on a body is equal to the change in the kinetic energy of the body

work done by net force $= \Delta E_k$

Example questions

Q5

A mass of 5.00 kg moving with an initial velocity of 12.0 m s^{-1} is brought to rest by a horizontal force over a distance of 12.0 m. What is the force?

Answer

The change in the kinetic energy of the mass is (final minus initial)

$$0 - \frac{1}{2}mv^2 = -\frac{1}{2} \times 5.00 \times 144$$
$$= -360\ \text{J}$$

The work done by the force f is

$$-fs = -12f$$

Hence

$$-12f = -360$$
$$\Rightarrow f = 30.0\ \text{N}$$

(There is a minus sign in the work done by f because the force is acting in a direction opposite to the motion and $\cos 180° = -1$.)

Q6

An electron is acted upon by an electric force that accelerates it from rest to a kinetic energy of 5.0×10^{-19} J. If this is done over a distance of 3.0 cm, find the electric force.

Answer

The work done by the electric force is the change in kinetic energy (the electric force is the only force acting and so it is the net force) and equals the product of force times distance. Hence

$$F \times 0.03 = 5.0 \times 10^{-19}$$
$$\Rightarrow F = 1.7 \times 10^{-17}\ \text{N}$$

Q7

A mass m hangs from two strings attached to the ceiling such that they make the same angle with the vertical (as shown in Figure 7.10). The strings are shortened very slowly so that the mass is raised a distance h above its original position. What is the work done by the tension in each string as the mass is so raised?

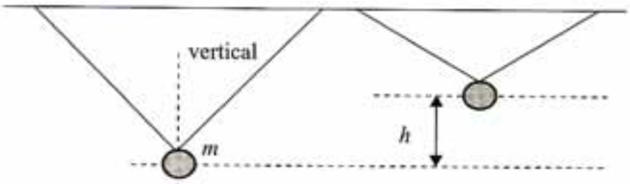

Figure 7.10.

Answer

The net work done is zero since the net force on the mass is zero. The work done by gravity is $-mgh$ and thus the work done by the two equal tension forces is $+mgh$. The work done by each is thus $mgh/2$.

Q8

A mass m hangs vertically at the end of a string of length L. A force F is applied to the mass horizontally so that it slowly moves to a position a distance h higher, as shown in Figure 7.11. What is the work done by the force F? (Note: F is not constant.)

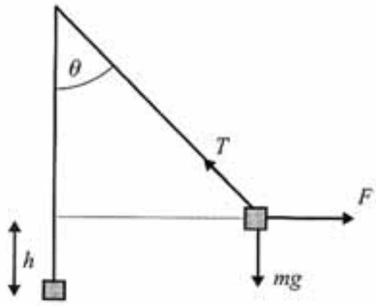

Figure 7.11.

Answer

The answer is obtained at once by noting that since the mass is in equilibrium at all times the net force is zero and hence the work done by the net force is zero. But T does zero work since it is always normal to the direction in which the mass moves (along the arc of a circle). The weight does work $-mgh$ and thus the work done by F must be $+mgh$.

Conservation of energy

We have already seen that, when a net force F performs work W on a body, then the kinetic energy of the body changes by W (here the subscripts 'i' and 'f' stand for 'initial' and 'final')

$$W = \Delta E_k$$
$$= \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

Consider now the case where the only force that does work on a body is gravity. This corresponds to motions in which the body is either in free fall (gravity is the only force acting) or the body is sliding on a frictionless surface (we now have the normal reaction force acting here as well as gravity but this force does zero work since it is normal to the direction of motion). Suppose that the vertical height of the body when the velocity is v_i is H, and the velocity becomes v_f at a height of h from the reference level (see Figure 7.12).

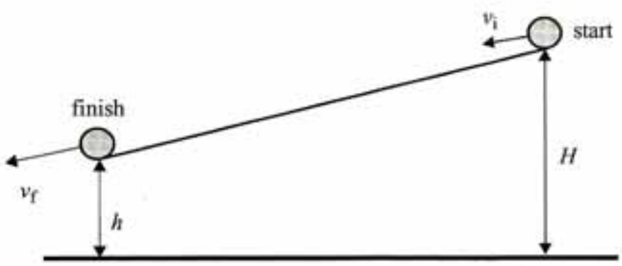

Figure 7.12 The total energy at the top and bottom (and at any point in between) of the incline is the same.

The work done by gravity is simply $mg(H-h)$ and so

$$mg(H-h) = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

This can be rearranged as

$$mgh + \frac{1}{2}mv_f^2 = mgH + \frac{1}{2}mv_i^2$$

which shows that in the motion of this body the sum of the kinetic and potential energies of the mass stays the same. Calling the quantity $mgh + \frac{1}{2}mv^2$ the total energy of the mass, then the result we derived states that the total energy E of the mass stays the same at all times, i.e. the total energy is conserved,

$$E_i = E_f$$

As the mass comes down the plane, its potential energy decreases but its kinetic energy increases in such a way that the sum stays the same. We have proven this result in the case in which gravity is the only force doing work in our problem. Consider now the following example questions.

Example questions

Q9

Find the speed of the mass at the end of a pendulum of length 1.00 m that starts from rest at an angle of 10° with the vertical.

Answer

Let us take as the reference level the lowest point of the pendulum (Figure 7.13). Then the total energy at that point is just kinetic, $E_k = \frac{1}{2}mv^2$, where v is the unknown speed. At the initial point the total energy is just potential, $E_p = mgh$, where

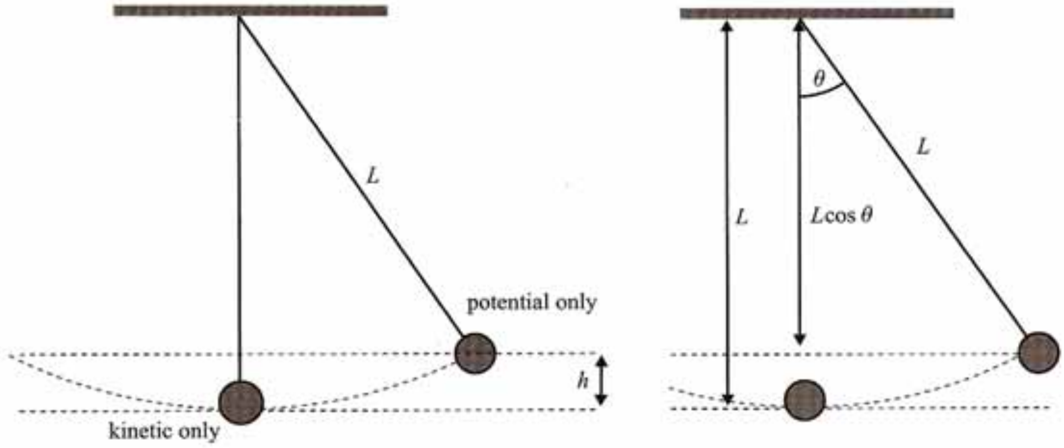

Figure 7.13.

h is the vertical difference in height between the two positions, that is

$$h = 1.00 - 1.00 \cos 10^\circ$$
$$= 0.015 \text{ m}$$

(see Figure 7.13)

Thus

$$\tfrac{1}{2}mv^2 = mgh$$
$$v = \sqrt{2gh}$$
$$= 0.55 \text{ m s}^{-1}$$

Note how the mass has dropped out of the problem. (At positions other than the two shown, the mass has both kinetic and potential energy.)

Q10

A mass rolls up an incline of angle 28° with an initial speed of 3.0 m s^{-1}. How far up the incline will the mass get?

Answer

Let the furthest the mass will get be a height h from the floor, as shown in Figure 7.14. At this point the kinetic energy must be zero since otherwise the mass would have climbed higher. Then the total energy at this point is just $E = mgh$. The total energy at the initial position is

$$E = \tfrac{1}{2}mv^2$$

and so

$$h = \frac{v^2}{2g}$$

giving $h = 0.45$ m. The distance moved along the plane is thus

$$\frac{0.45}{\sin 28^\circ} = 0.96 \text{ m}.$$

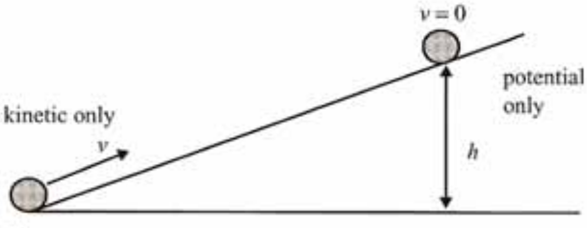

Figure 7.14.

Q11

What must the minimum speed of the mass in Figure 7.15 be at the initial point such that the mass makes it over the barrier of height h?

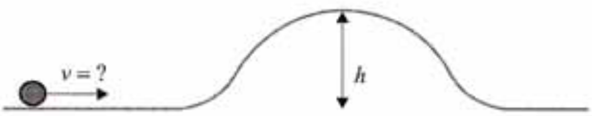

Figure 7.15.

Answer

To make it over the barrier the mass must be able to reach the highest point. Any speed it has there will then make it roll over. Thus, at the very least, we must be able to get the ball to the highest point with zero speed. Then the total energy would be just $E = mgh$ and the total energy at the starting position is $E = \tfrac{1}{2}mv^2$. Thus, the speed must be bigger than $v = \sqrt{2gh}$. Note that if the initial speed u of the mass is larger than $v = \sqrt{2gh}$, then when the mass makes it to the other side of the barrier its speed will be the same as the starting speed u.

Q12

A mass rolls off a 1.0 m high table with a speed of 4.0 m s^{-1}, as shown in Figure 7.16. With what speed does it strike the floor?

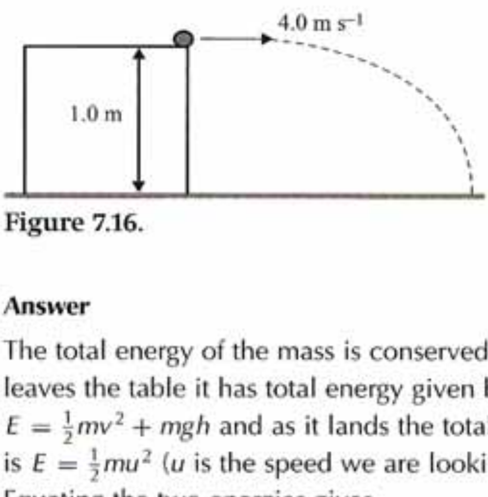

Figure 7.16.

Answer

The total energy of the mass is conserved. As it leaves the table it has total energy given by $E = \tfrac{1}{2}mv^2 + mgh$ and as it lands the total energy is $E = \tfrac{1}{2}mu^2$ (u is the speed we are looking for). Equating the two energies gives

$$\tfrac{1}{2}mu^2 = \tfrac{1}{2}mv^2 + mgh$$
$$\Rightarrow u^2 = v^2 + 2gh$$
$$= 16 + 20 = 36$$
$$\Rightarrow u = 6.0 \text{ m s}^{-1}$$

Q13

A ball is thrown vertically upward with a speed of $4.0\ \text{m s}^{-1}$ from a height of 1.0 m from the floor, as shown in Figure 7.17. With what speed does the ball strike the floor?

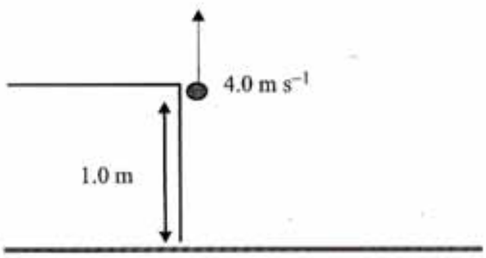

Figure 7.17.

Answer

Working in precisely the same way as in the previous example we find

$$\tfrac{1}{2}mu^2 = \tfrac{1}{2}mv^2 + mgh$$
$$\Rightarrow u^2 = v^2 + 2gh$$
$$= 16 + 20$$
$$= 36$$
$$\Rightarrow u = 6.0\ \text{m s}^{-1}$$

(The answer is the same as that for example question 12 – why?)

If, in addition to the weight, there are spring tension forces acting in our system, then the previous discussion generalizes to again lead to

$$E_i = E_f$$

where now the total mechanical energy includes elastic potential energy as well, that is

$$E = \tfrac{1}{2}mv^2 + mgh + \tfrac{1}{2}kx^2$$

Example question

Q14

A body of mass 0.40 kg is held next to a compressed spring as shown in Figure 7.18. The spring constant is $k = 250\ \text{N m}^{-1}$ and the compression of the spring is 12 cm. The mass is then released. Find the speed of the body when it is at a height of 20 cm from the horizontal.

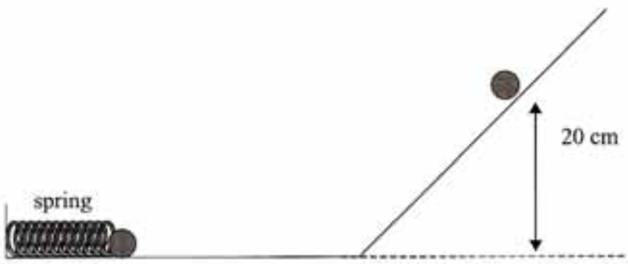

Figure 7.18.

Answer

Initially the total mechanical energy of the system is only the elastic potential energy of the spring

$$E = \tfrac{1}{2}kx^2$$
$$= \tfrac{1}{2} \times 250 \times 0.12^2$$
$$= 1.8\ \text{J}$$

At a height of 20 cm from the floor, the total mechanical energy consists of kinetic and gravitational energies only. The spring has decompressed and so has no elastic potential energy. Then

$$E = \tfrac{1}{2}mv^2 + mgh$$
$$= \tfrac{1}{2}(0.4)v^2 + (0.4)(10)(0.20)$$
$$= 0.2v^2 + 0.8$$

Thus

$$0.2v^2 + 0.8 = 1.8$$
$$\Rightarrow 0.2v^2 = 1.0$$
$$\Rightarrow v^2 = 5$$
$$\Rightarrow v = 2.2\ \text{m s}^{-1}$$

Frictional forces

In the presence of friction and other resistance forces, the *mechanical energy* of a system (i.e. the sum of kinetic, gravitational potential and elastic potential energies) will not be conserved. These forces will, in general, decrease the total mechanical energy of the system. Similarly, external forces, such as forces due to engines, may increase the mechanical energy of a system. In these cases we may write

$$W = \Delta E$$

where W stands for the total work done by the *external forces* and ΔE is the change in the mechanical energy of the system. By external forces we mean forces *other* than weight and spring tension forces.

This equation is easily understood in the following way. If there are no external forces, then $W = 0$, $\Delta E = 0$ and the total mechanical energy stays the same: it is conserved.

If, on the other hand, external forces do act on the system, then the work they do goes into changing the mechanical energy. If the work done is negative (resistance forces), the mechanical energy decreases. If the work done is positive (pulling forces), the mechanical energy increases.

Example question

Q15

A body of mass 2.0 kg (initially at rest) slides down a curved path of total length 16 m as shown in Figure 7.19. When it reaches the bottom, its speed is measured and found to equal 6.0 m s^{-1}. Show that there is a force resisting the motion. Assuming the force to have constant magnitude, determine what that magnitude is.

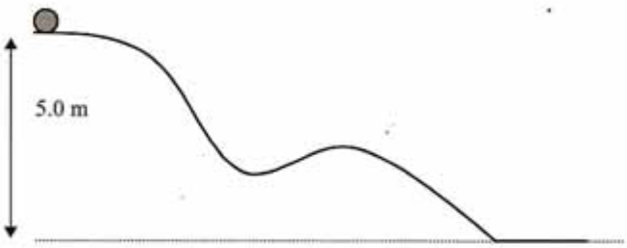

Figure 7.19.

Answer

Without any resistance forces, the speed at the bottom is expected to be

$$v = \sqrt{2gh} = \sqrt{2 \times 10 \times 5.0} = 10\ \text{m s}^{-1}$$

The measured speed is less than this and so there is a resistance force. The total mechanical energy at the top is

$$E_{\text{top}} = mgh = 2.0 \times 10 \times 5.0 = 100\ \text{J}$$

At the bottom it is

$$E_{\text{bottom}} = \tfrac{1}{2}mv^2 = \tfrac{1}{2} \times 2.0 \times (6.0)^2 = 36\ \text{J}$$

The total energy decreased by 64 J – this must be the work done by the resistance force. Thus

$$fs = 64\ \text{J}$$
$$\Rightarrow f = \frac{64}{16} = 4.0\ \text{N}$$

We have seen that in the presence of external forces the total mechanical energy of a system is not conserved. The change in the total energy is the work done by the external forces. Another way of looking at this is to say the change in the mechanical energy has gone into other forms of energy *not included* in the mechanical energy, such as thermal energy ('heat') and sound. In this way *total energy* (which now includes the other forms as well as the mechanical energy) *is* conserved. This is the general form of the law of conservation of energy, one of the most important principles of physics.

▶ The total energy cannot be created or destroyed but can only be transformed from one form into another.

Power

When a machine performs work, it is important to know not only how much work is being done but also how much work is performed within a given time interval. A cyclist performs a lot of work in a lifetime of cycling, but the same work is performed by a powerful car engine in a much shorter time. **Power** is the rate at which work is being performed.

▶ When a quantity of work ΔW is performed within a time interval Δt the ratio

$$P = \frac{\Delta W}{\Delta t}$$

is called the power developed. Its unit is joule per second and this is given the name watt (W): $1\ \text{W} = 1\ \text{J s}^{-1}$.

Another common unit for power when it comes to machines and car engines is the horsepower, hp, a non-SI unit that equals 746 W.

Consider a constant force F, which acts on a body of mass m. The force does an amount of work $F\,\Delta x$ in moving the body a small distance Δx along its direction. If this work is performed in time Δt, then

$$\begin{aligned} P &= \frac{\Delta W}{\Delta t} \\ &= F\,\frac{\Delta x}{\Delta t} \\ &= F v \end{aligned}$$

where v is the instantaneous speed of the mass. This is the power produced in making the body move at speed v. As the speed increases, the power necessary increases as well. Consider an aeroplane moving at constant speed on a straight-line path. If the power produced by its engines is P, and the force pushing it forward is F, then P, F and v are related by the equation above. But since the plane moves with no acceleration, the total force of air resistance must equal F. Hence the force of air resistance can be found simply from the power of the plane's engines and the constant speed with which it coasts.

Example questions

Q16

What is the minimum power required to lift a mass of 50.0 kg up a vertical distance of 12 m in 5.0 s?

Answer

The work performed to lift the mass is

$$\begin{aligned} mgh &= 50.0 \times 10 \times 12 \\ &= 6.0 \times 10^3\ \text{J} \end{aligned}$$

The power is thus

$$\frac{6.0 \times 10^3}{5.0} = 1200\ \text{W}$$

This is only the minimum power required. In practice, the mass has to be accelerated from rest, which will require additional work and hence more power.

Q17

A helicopter rotor whose length is R pushes air downwards with a speed v. Assuming that the density of air is constant and equals ρ and the mass of the helicopter is M, find v. You may assume that the rotor forces the air in a circle of radius R (spanned by the rotor) to move with the downward speed v. Hence find the power developed by the engine. How does this power depend on the linear size of the helicopter?

Answer

The momentum of the air under the rotor is mv, where m is the mass of air in a circle of radius R. In time Δt the mass is enclosed in a cylinder of radius R and height $v\Delta t$. Thus, the momentum of this mass is $\rho\pi R^2 v^2 \Delta t$ and its rate of change is $\rho\pi R^2 v^2$. This is the force on the helicopter upwards, which must equal the helicopter's weight of Mg. Thus

$$\begin{aligned} Mg &= \rho\pi R^2 v^2 \\ \Rightarrow v &= \sqrt{\frac{Mg}{\rho\pi R^2}} \end{aligned}$$

The power required from the helicopter engine is thus

$$\begin{aligned} P &= F v \\ &= Mg\sqrt{\frac{Mg}{\rho\pi R^2}} \end{aligned}$$

To find the dependence on a typical linear size L of the helicopter, note that the weight depends on L as L^3 and so

$$P \propto L^3\sqrt{\frac{L^3}{L^2}} \propto L^{7/2}$$

This implies that if the length of a helicopter is 16 times that of a model helicopter, its required power will be $16^{7/2} \approx 16\,000$ times larger than that for the model.

Efficiency

Suppose that a mass is being pulled up along a rough inclined plane with *constant* speed. Let the mass be 15 kg and the angle of the incline 45°. The constant frictional force opposing the motion is 42 N.

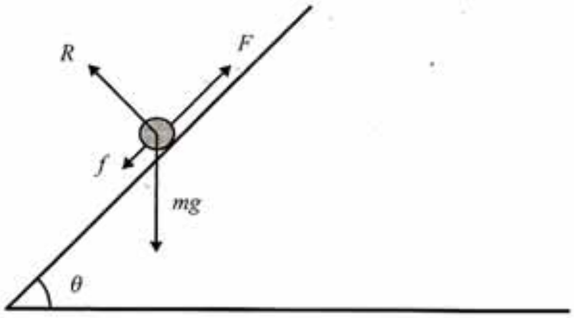

Figure 7.20.

The forces on the mass are shown in Figure 7.20 and we know that

$$\begin{aligned} R &= mg\cos\theta \\ &= 106.1\ \text{N} \end{aligned}$$

$$\begin{aligned} F &= mg\sin\theta + f \\ &= 106.1 + 42 \\ &= 148.1\ \text{N} \approx 150\ \text{N} \end{aligned}$$

since the mass has no acceleration. Let the force raise the mass a distance of 20 m along the plane. The work done by the force F is

$$\begin{aligned} W &= 148.1 \times 20 \\ &= 2960\ \text{J} \approx 3.0 \times 10^3\ \text{J} \end{aligned}$$

The force effectively raised the 15 kg a vertical height of 14.1 m and so increased the potential energy of the mass by $mgh = 2121$ J. The efficiency with which the force raised the mass is thus

$$\begin{aligned} \eta &= \frac{\text{useful work}}{\text{actual work}} \\ &= \frac{2121}{2960} \\ &= 0.72 \end{aligned}$$

Example question

Q18

A 0.50 kg battery-operated toy train moves with constant velocity 0.30 m s^{-1} along a level track. The power of the motor in the train is 2.0 W and the total force opposing the motion of the train is 5.0 N.

(a) What is the efficiency of the train's motor?

(b) Assuming the efficiency and the opposing force stay the same, calculate the speed of the train as it climbs an incline of 10.0° to the horizontal.

Answer

(a) The power delivered by the motor is 2.0 W. Since the speed is constant, the force developed by the motor is also 5.0 N. The power used in moving the train is $Fv = 5.0 \times 0.30 = 1.5$ W. Hence the efficiency is

$$\begin{aligned} \eta &= \frac{1.5\ \text{W}}{2.0\ \text{W}} \\ &= 0.75 \end{aligned}$$

(b) The net force pushing the train up the incline is

$$\begin{aligned} F &= mg\sin\theta + 5.0 \\ &= 0.50 \times 10 \times \sin 10° + 5.0 \\ &= 5.89\ \text{N} \approx 5.9\ \text{N} \end{aligned}$$

Thus

$$\begin{aligned} \eta &= \frac{5.89 \times v}{2.0\ \text{W}} \\ &= 0.75 \\ \Rightarrow v &= \frac{2.0 \times 0.75}{5.89} \\ &= 0.26\ \text{m s}^{-1} \end{aligned}$$

Kinetic energy and momentum

We have seen in Chapter 2.6 on momentum that, in a collision or explosion where no external forces are present, the total momentum of the system is conserved. You can easily convince yourself that in the three collisions illustrated in Figure 7.21 momentum is conserved. The incoming body has mass 8.0 kg and the other a mass of 12 kg.

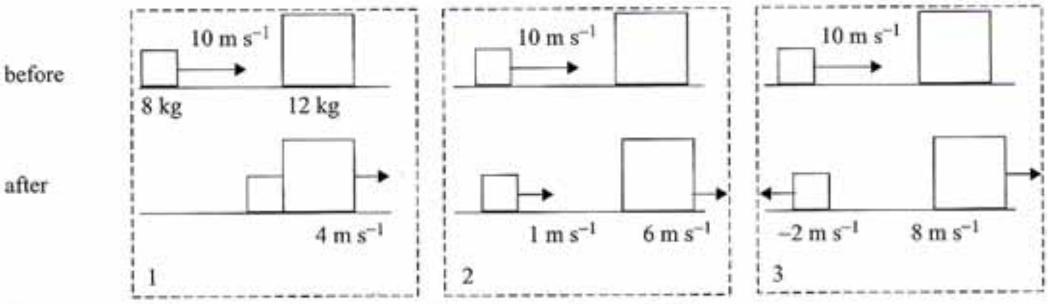

Figure 7.21 Momentum is conserved in these three collisions.

In the first collision the bodies have stuck together and move as one. In the second the incoming body has slowed down as a result of the collision and the heavy body moves faster. In the third the incoming body has bounced back.

Let us examine these collisions from the point of view of energy.

In all cases the total kinetic energy before the collision is

$$E_k = \frac{1}{2} \times 8 \times 10^2 = 400\,\text{J}$$

The total kinetic energy after the collision in each case is:

case 1 $E_k = \frac{1}{2} \times 20 \times 4^2 = 160\,\text{J}$

case 2 $E_k = \frac{1}{2} \times 8 \times 1^2 + \frac{1}{2} \times 12 \times 6^2 = 220\,\text{J}$

case 3 $E_k = \frac{1}{2} \times 8 \times 2^2 + \frac{1}{2} \times 12 \times 8^2 = 400\,\text{J}$

We thus observe that *whereas momentum is conserved in all cases,* kinetic energy is not. When kinetic energy is conserved (case 3), the collision is said to be *elastic.* When it is not (cases 1 and 2), the collision is *inelastic.* In an inelastic collision, kinetic energy is lost. When the bodies stick together after a collision (case 1), the collision is said to be *totally inelastic* and in this case the maximum possible kinetic energy is lost.

The lost kinetic energy gets transformed into other forms of energy, such as thermal energy, deformation energy (if the bodies are permanently deformed as a result of the collision) and sound energy.

Example questions

Q19

A moving body of mass m collides with a stationary body of double the mass and sticks to it. What fraction of the original kinetic energy is lost?

Answer

The original kinetic energy is $\frac{1}{2}mv^2$, where v is the speed of the incoming mass. After the collision the two bodies move as one with speed u that can be found from momentum conservation:

$$mv = (m + 2m)u$$
$$\Rightarrow u = \frac{v}{3}$$

The total kinetic energy after the collision is therefore

$$\frac{1}{2}(3m)\left(\frac{v}{3}\right)^2 = \frac{mv^2}{6}$$

and so the lost kinetic energy is

$$\frac{mv^2}{2} - \frac{mv^2}{6} = \frac{mv^2}{3}$$

The fraction of the original energy that is lost is thus

$$\frac{mv^2/3}{mv^2/2} = \frac{2}{3}$$

Q20

A body at rest of mass M explodes into two pieces of masses $M/4$ and $3M/4$. Calculate the ratio of the kinetic energies of the two fragments.

Answer

Here it pays to derive a very useful expression for kinetic energy in terms of momentum. Since

$$E_k = \frac{mv^2}{2}$$

it follows that

$$E_k = \frac{mv^2}{2} \times \frac{m}{m} = \frac{m^2v^2}{2m} = \frac{p^2}{2m}$$

The total momentum before the explosion is zero, so it is zero after as well. Thus, the two fragments must have *equal and opposite momenta*. Hence

$$\frac{E_{light}}{E_{heavy}} = \frac{p^2/(2M_{light})}{p^2/(2M_{heavy})} = \frac{M_{heavy}}{M_{light}} = \frac{3M/4}{M/4} = 3$$

The problem of least time

In Figure 7.22 a number of paths join the starting position A to the final position B.

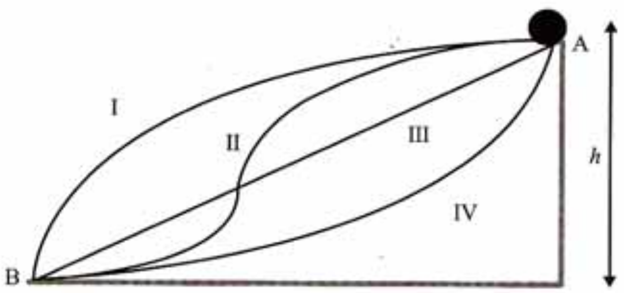

Figure 7.22.

A mass m at A will start with a tiny speed and move down to B. As we saw, the speed at B will be the same no matter what path the mass follows. The speed will equal $\sqrt{2gh}$ in all cases. This does not mean, however, that the time taken is the same for all paths. Finding the path joining A to B such that a mass takes the least time is a famous problem in physics and requires the development of a branch of calculus called the calculus of variations. It is called the brachistochrone (least time) problem. The answer is that the curve joining A to B must be a cycloid. This is the curve traced out by a point on the rim of a wheel as the wheel rolls. In Figure 7.22, curve IV resembles a cycloid most. This problem was posed to both Newton and Leibniz (the inventors of calculus) by the Swiss mathematician Bernoulli. When Bernoulli saw the solutions given by the two men, he is supposed to have said of Newton 'one can always tell a lion by its claws'.

Questions

1 A horizontal force of 24 N pulls a body a distance of 5.0 m along its direction. Calculate the work done by the force.

2 A block slides along a rough table and is brought to rest after travelling a distance of 2.4 m. The frictional force is assumed constant at 3.2 N. Calculate the work done by the frictional force.

3 A block is pulled as shown in Figure 7.23 by a force making an angle of 20° to the horizontal. Find the work done by the pulling force when its point of application has moved 15 m.

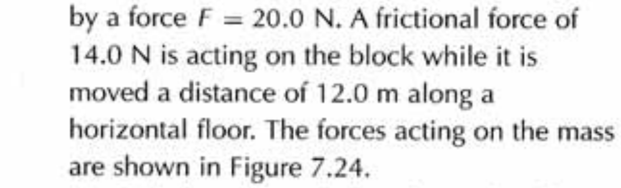

Figure 7.23 For question 3.

4 A block of mass 4.0 kg is pushed to the right by a force $F = 20.0$ N. A frictional force of 14.0 N is acting on the block while it is moved a distance of 12.0 m along a horizontal floor. The forces acting on the mass are shown in Figure 7.24.

(a) Calculate the work done by each of the four forces acting on the mass.

(b) Hence find the net work done.

(c) By how much does the kinetic energy of the mass change?

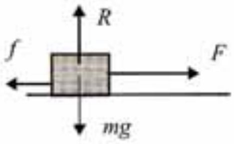

Figure 7.24 For question 4.

5 A weight lifter slowly lifts a 100 kg mass from the floor up a vertical distance of 1.90 m and then slowly lets it down to the floor again.
(a) Find the work done by the weight of the mass on the way up.
(b) Find the work done by the force exerted by the weight lifter when lifting the weight up.
(c) What is the total work done by the weight on the way up and the way down?

6 A block of mass 2.0 kg and an initial speed of 5.4 m s^{-1} slides on a rough horizontal surface and is eventually brought to rest after travelling a distance of 4.0 m. Calculate the frictional force between the block and the surface.

7 A spring of spring constant $k = 200$ N m^{-1} is slowly extended from an extension of 3.0 cm to an extension of 5.0 cm. How much work is done by the extending force?

8 A spring of spring constant $k = 150$ N m^{-1} is compressed by 4.0 cm. The spring is horizontal and a mass of 1.0 kg is held to the right end of the spring. If the mass is released, with what speed will it move away?

9 Look at Figure 7.25.
(a) What is the minimum speed v the mass must have in order to make it to position B? What speed will the mass have at B?
(b) If $v = 12.0$ m s^{-1}, what will the speed be at A and B?

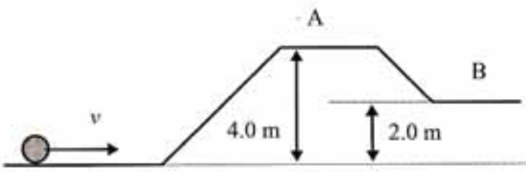

Figure 7.25 For question 9.

10 A mass is released from rest from the position shown in Figure 7.26. What will its speed be as it goes past positions A and B?

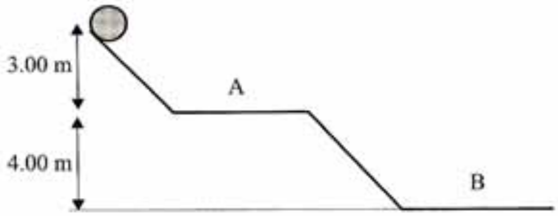

Figure 7.26 For question 10.

11 The speed of the 8.0 kg mass in position A in Figure 7.27 is 6.0 m s^{-1}. By the time it gets to B the speed is measured to be 12.0 m s^{-1}.

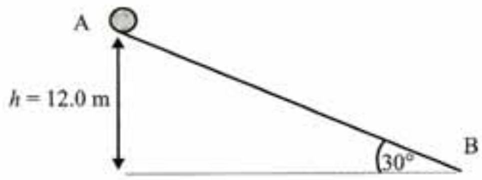

Figure 7.27 For question 11.

What is the frictional force opposing the motion? (The frictional force is acting along the plane.)

12 A toy gun shoots a 20.0 g ball when a spring of spring constant 12.0 N m^{-1} decompresses. The amount of compression is 10.0 cm (see Figure 7.28). With what speed does the ball exit the gun, assuming that there is no friction between the ball and the gun? If, instead, there is a frictional force of 0.05 N opposing the motion of the ball, what will the exit speed be in this case?

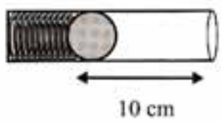

Figure 7.28 For question 12.

13 A variable force F acts on a body of mass $m = 2.0$ kg initially at rest, moving it along a straight horizontal surface. For the first 2.0 m the force is constant at 4.0 N. In the next 2.0 m it is constant at 8.0 N. In the next 2.0 m it drops from 8.0 N to 2.0 N uniformly. It then increases uniformly from 2.0 N to 6.0 N in the next 2.0 m. It then remains constant at 6.0 N for the next 4.0 m.

(a) Draw a graph of the force versus distance.
(b) Find the work done by this force.
(c) What is the final speed of the mass?

14 A body of mass 12.0 kg is dropped vertically from rest from a height of 80.0 m. Ignoring any resistance forces during the motion of this body, draw graphs to represent the variation with distance fallen of

(a) the potential energy;
(b) the kinetic energy.

For the same motion draw graphs to represent the variation with time of

(c) the potential energy;
(d) the kinetic energy.
(e) Describe qualitatively the effect of a constant resistance force on each of the four graphs you drew.

15 A 25.0 kg block is very slowly raised up a vertical distance of 10.0 m by a rope attached to an electric motor in a time of 8.2 s. What is the power developed in the motor?

16 The engine of a car is developing a power of 90.0 kW when it is moving on a horizontal road at a constant speed of 100.0 km h^{-1}. What is the total horizontal force opposing the motion of the car?

17 The motor of an elevator develops power at a rate of 2500 W.

(a) At what speed can a 1200 kg load be raised?
(b) In practice it is found that the load is lifted more slowly than indicated by your answer to (a). Suggest reasons why this is so.

18 A load of 50.0 kg is raised a vertical distance of 15 m in 125 s by a motor.

(a) What is the power necessary for this?
(b) The power supplied by the motor is in fact 80 W. Calculate the efficiency of the motor.
(c) If the same motor is now used to raise a load of 100.0 kg and the efficiency remains the same, how long would that take?

19 For cars having the same shape but different size engines it is true that the power developed by the car's engine is proportional to the third power of the car's maximum speed. What does this imply about the speed dependence of the wind resistance force?

20 The top speed of a car whose engine is delivering 250 kW of power is 240 km h^{-1}. Calculate the value of the resistance force on the car when it is travelling at its top speed on a level road.

21 Describe the energy transformations taking place when a body of mass 5.0 kg:

(a) falls from a height of 50 m without air resistance;
(b) falls from a height of 50 m with constant speed;
(c) is being pushed up an incline of 30° to the horizontal with constant speed.

22 An elevator starts on the ground floor and stops on the 10th floor of a high-rise building. The elevator picks up a constant speed by the time it reaches the 1st floor and decelerates to rest between the 9th and 10th floors. Describe the energy transformations taking place between the 1st and 9th floors.

23 A car of mass 1200 kg starts from rest, accelerates uniformly to a speed of 4.0 m s^{-1} in 2.0 s and continues moving at this constant speed in a horizontal straight line for an additional 10 s. The brakes are then applied and the car is brought to rest in 4.0 s. A constant resistance force of 500 N is acting on the car during its entire motion.

(a) Calculate the force accelerating the car in the first 2.0 s of the motion.
(b) Calculate the average power developed by the engine in the first 2.0 s of the motion.
(c) Calculate the force pushing the car forward in the next 10 s.
(d) Calculate the power developed by the engine in those 10 s.
(e) Calculate the braking force in the last 4.0 s of the motion.
(f) Describe the energy transformations that have taken place in the 16 s of the motion of this car.

24 A mass of 6.0 kg moving at 4.0 m s^{-1} collides with a mass of 8.0 kg at rest on a frictionless surface and sticks to it. How much kinetic energy was lost in the collision?

25 Two masses of 2.0 kg and 4.0 kg are held in place, compressing a spring between them. When they are released, the 2.0 kg moves away with a speed of 3.0 m s^{-1}. What was the energy stored in the spring?

26 A block of mass 0.400 kg is kept in place so it compresses a spring of spring constant 120 N m^{-1} by 15 cm (see Figure 7.29). The block rests on a rough surface and the frictional force between the block and the surface when the block begins to slide is 1.2 N.

Figure 7.29 For question 26.

(a) What speed will the block have when the spring returns to its natural length?
(b) What percentage is this of the speed the mass would have had in the absence of friction?

27 Two bodies are connected by a string that goes over a pulley, as shown in Figure 7.30. The lighter body is resting on the floor and the other is being held in place a distance of 5.0 m from the floor. The heavier body is then released. Calculate the speeds of the two bodies as the heavy mass is about to hit the floor.

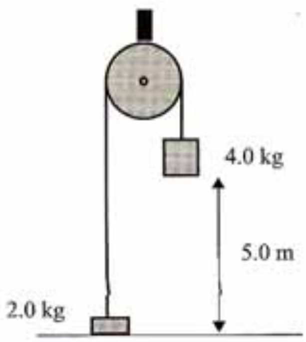

Figure 7.30 For question 27.

28 A mass m of 4.0 kg slides down a frictionless incline of $\theta = 30°$ to the horizontal.
(a) Plot a graph of the kinetic and potential energies of the mass as a function of time.
(b) Plot a graph of the kinetic and potential energies of the mass as a function of distance travelled along the incline.
(c) On each graph, plot the sum of the potential and kinetic energies. The mass starts from rest from a height of 20 m.

29 Show that an alternative formula for kinetic energy is $E_k = \frac{p^2}{2m}$, where p is the momentum of the mass m. This is very useful when dealing with collisions.

30 A body of mass M, initially at rest, explodes and splits into two pieces of mass $M/3$ and $2M/3$, respectively. Find the ratio of the kinetic energies of the two pieces. (Use the formula from the previous problem.)

31 A mass m is being pulled up an inclined plane of angle θ by a rope along the plane.
(a) What is the tension in the rope if the mass moves up at constant speed v?
(b) What is the work done by the tension when the mass moves up a distance of d m along the plane?
(c) What is the work done by the weight of the mass?
(d) What is the work done by the normal reaction force on the mass?
(e) What is the net work done on the mass?

HL only

32 A mass $m = 2.0$ kg is attached to the end of a string of length $L = 4.5$ m. The other end of the string is attached to the ceiling. The string is displaced from the vertical by an angle $\theta_0 = 50°$ and then released. What is the tension in the string when the string makes an angle $\theta = 20°$ with the vertical?

33 A car of mass 1200 kg is moving on a horizontal road with constant speed 30 m s^{-1}. The engine is then turned off and the car will eventually stop under the action of an air resistance force. Figure 7.31 shows the variation of the car's speed with time after the engine has been turned off.

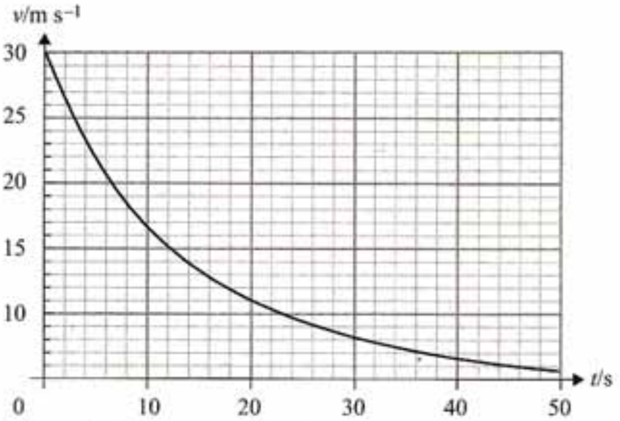

Figure 7.31 For question 33.

(a) Calculate the average acceleration of the car in the first and second 10 s intervals.

(b) Explain why it takes longer to reduce the speed from 20.0 $m\,s^{-1}$ to 10.0 $m\,s^{-1}$ compared with from 30.0 $m\,s^{-1}$ to 20.0 $m\,s^{-1}$.

(c) The average speed in the first 10 s interval is 21.8 $m\,s^{-1}$ and in the second it is 13.5 $m\,s^{-1}$. Use this information and your answer in (a) to deduce that the air resistance force is proportional to the square of the speed.

(d) Calculate the distance travelled by the car in the first and second 10 s intervals.

(e) Calculate the work done by the resistance force in the first and second 10 s intervals.

34 A bungee jumper of mass 60 kg jumps from a bridge 24 m above the surface of the water. The rope is 12 m long and is assumed to obey Hooke's law.

(a) What should the spring constant of the rope be if the woman is to just reach the water?

(b) The same rope is used by a man whose mass is more than 60 kg. Explain why the man will not stop before reaching the water. (Treat the jumper as a point and ignore any resistance to motion.)

35 For the bungee jumper of mass 60 kg in question 34, calculate:

(a) the speed of the jumper when she has fallen by 12 m;

(b) the maximum speed attained by the jumper during her fall.

(c) Explain why the maximum speed is reached after falling more than a distance of 12 m (the unstretched length of the rope).

HL only

(d) Sketch a graph to show the variation of the speed of the jumper with distance fallen.

36 A carriage of mass 800 kg moving at 5.0 $m\,s^{-1}$ collides with another carriage of mass 1200 kg that is initially at rest. Both carriages are equipped with buffers. The graph in Figure 7.32 shows the velocities of the two carriages before, during and after the collision.

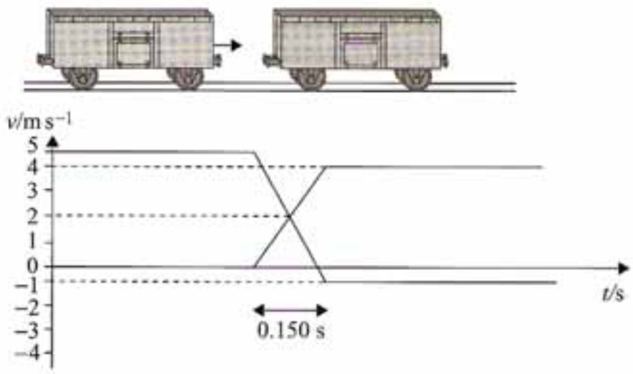

Figure 7.32 For question 36.

Use the graph to:

(a) show that the collision has been elastic;

(b) calculate the average force on each carriage during the collision;

(c) calculate the impulse given to the heavy carriage.

(d) If the buffers on the two carriages had been stiffer, the time of contact would have been less but the final velocities would be unchanged. How would your answers to (b) and (c) change?

(e) Calculate the kinetic energy of the two carriages at the time during the collision when both have the same velocity and compare your answer with the final kinetic energy of the carriages. How do you account for the difference?